Some Fundamental Problems of Opinion Modeling with Implications to Committee Composition and Social Choice

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***Abstract*—The standard choice theory's assumption of transitive preference relation is discussed. It is argued that in multi-criterion contexts it may be too demanding. The spatial choice theory's assumption that individuals prefer the alternative closest to their optimum point is also called into question in multi-criterion settings. Fuzzy preference relations may suggest avenues to overcome these and other problems. Especially in setting up multi-member representative bodies the fuzzy preference models may turn out useful once the problems of opinion elicitation are solved.**

I. Introduction

Then standard assumption in the modern theory of individual and group decision making is that the individuals are endowed with complete (connected) and transitive preference relations over the decision alternatives. Assuming, moreover, that for any fixed alternative x the individual deems those alternatives that are no better (worse, respectively) than x – the inferior (superior) set of x – as forming a closed set, it possible to prove that the preference relation can be represented by a utility function. Hence, if the individual is rational in the obvious sense of acting according to her preferences, then she ipso facto acts as if she were maximizing her utility [8]. Indeed, with analogous assumptions similar and even stronger representation theorems can be proven for choices under risk (lottery choice) and uncertainty (betting).

The axiomatic theory of choice has also been extended to the domain of group choice. There the representation theorems are of secondary significance. Rather the focus has been on the outcomes of preference aggregation procedures. In other words, the attention has been directed towards systems transforming sets of individuals opinions into collective decisions. There various anomalies have been encountered. The best-known of them is Arrow's impossibility theorem that dashes the hopes of constructing a aggregation method that is always in an agreement with a set of intuitively plausible and innocent-looking criteria regarding the relationship between decision outcomes and the expressed opinions [1]. A host of similar incompatibility results involving various desiderata have subsequently been proven in the literature [13]. In fact, the social choice theory has become notorious for these basically negative results.

This paper deals with the fundamental assumption underlying all these negative results, viz. that individuals are characterized by complete and transitive preferences relations or rankings. The next section purports to show that intransitive individual preference makes perfect sense in some circumstances. The next section focuses on preferences that have spatial representations. These play a prominent role in modern social choice theory. We show that the assumptions under which preferences can be spatially represented are serious and often violated. But are there alternatives to the ranking assumption? There are, most notably fuzzy preferences. When these are available, several paradoxes can be avoided. Making reasonable choices under fuzziness still requires that one's ideas are fixed with regard to the desiderata of social choices. Especially, one should make up one's mind with regard to the binary vs. positional winning intuitions. Once a stand on this traditional issue has been taken, both social choices and committee composition turn out to be relatively straight-forward.

II. Cyclical individual preference

The illustrate the possibility of a cyclic individual preference relation consider the choice setting the U.S. voters were faced with in the 2000 presidential election. There were three main contestants: Bush, Gore and Nader. Suppose that the voter considers three main policy issues: environmental policy, employment policy and crime prevention policy. Suppose, moreover, that she deems these three policy domains of roughly equal importance at least to the extent that any two criteria are together more important than the third alone.

Her views on the candidates' rankings over these three criteria might look like the following (Table I).

Under the assumptions made above, the individual has obviously a cyclic preference relation over the $\{Bush, Gore, Nader\}$: Bush is preferred to Nader, Gore preferred to Bush, and Nader preferred to Gore.

The standard argument in defence of transitive preferences is what is known as the money pump. A person with cyclic preferences can lose all her money by first giving her an

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environment	employment	crime prevention
Nader	Gore	Bush
Gore	Bush	Nader
Bush	Nader	Gore

TABLE I
CYCLIC PREFERENCE RELATION

issue	*issue 1*	*issue 2*	*issue 3*	*votes*
voter A	X	X	Y	X
voter B	X	Y	X	X
voter C	Y	X	X	X
voter D	Y	Y	Y	Y
voter E	Y	Y	Y	Y
winner	Y	Y	Y	?

TABLE II
OSTROGORSKI'S PARADOX

alternative and then offering her – for a small sum – the alternative that she prefers to the first one. Once she has paid the sum and received the preferred alternative, she is again offered – for a small sum – another alternative that she prefers to the previous one. If the price is small enough the individual should accept those offers (that's what preference is all about, isn't it?), whereupon she is offered – again for a small price – the alternative from which the process started. Now we are back where we started from, only the individual has paid three small sums to get there. And the process can be repeated ad libitum.

The money pump is a strong argument for transitive preferences. The point of the above example is to say that cyclic preferences are not always unreasonable. Experimental evidence suggests that they are rather common in settings involving lotteries [24].

III. SPATIAL REPRESENTATION OF PREFERENCES

Suppose that in an election there are 5 voters, 2 parties and 3 issues. Suppose, moreover, that each voter considers these issues to be of equal importance and that there are no other considerations in their mind that would determine their opinion on the parties. Consider two ways of determining the election result. (1) Each voter votes for the party that is closer to her opinion on more issues than the other party and whichever party gets more votes than the other is the winner. (2) For each issue the winner is the party that gets more votes than its competitor and the election winner is the party winning on more issues than the other. In a nutshell, Ostrogorski's paradox occurs when the election result differs in these two cases. Consider the following distribution of opinions on parties X and Y (Table 2).

This is a rather strong version of the paradox since not only are the results different under procedures (1) and (2), but the winner under (2) is a unanimous one. Replacing any one Y with an X in the table would result in a weaker version of the paradox where "just" a majority winner is different under (1) and (2).

Replace now "voter" with "criterion" throughout in the preceding table and consider the procedure of forming an individual preference over two candidates X and Y. For example, in political competition the criteria could be relevant educational background, political experience, negotiation skills in the issue at hand, relevant political connections, etc. The issues might be e.g. education, economy and foreign policy. Each entry in the table then indicates which alternative is better on the criterion represented by the row when the issue is the one represented by the column. Suppose that the criterion-wise preference is formed on the basis of which alternative is better on more issues than the other. If all issues and criteria are deemed of equal importance, the decision of which candidate the individual should vote is ambiguous: the row-column aggregation with the majority principle suggests X, but the column-row aggregation with the same principle yields Y.

Suppose now that the issues span a 3-dimensional Euclidean space where X and Y are located as two distinct points. The individual whose views are represented in the above table would then be located in this space so that on each dimension her ideal point (i.e. the point that represents her) is closer to Y than to X. However, it cannot be inferred on this basis alone that in a pairwise comparison between X and Y she would vote for Y. In fact, if she resorts to the wholly reasonable principle of basing her choice on the criterion-wise performance of candidates, she will vote for X. After all, X outperforms Y on three criteria, while Y beats X on only two.

It is worth pointing out that the problem here cannot be resolved by assigning salience weights to issue dimensions, since Y is closer to the individual on each dimension. Strategic considerations – which of course may underly occasional votes against preferences – do not enter into the calculus dictating the choice of X rather than Y since the the agenda consists of only two alternatives and the ideal points of other voters are not known.

IV. FUZZY SOCIAL CHOICE

The study of voting procedures is at the hearth of democratic theory and, thus, occupies an important place in modern science of politics [19]. The theoretical background of this work lies in social choice theory. Some aspects of this theory have also been approached from the angle of fuzzy sets [17], [11], [12].

Consider a non-fuzzy set X of decision alternatives (candidates, policies, other entities of value). Then the fuzzy m-ary relation R over X can be defined using the membership function μ_R as follows:

$$\mu_R : X^m \to [0,1] \tag{1}$$

with X^m denoting the Cartesian product set of X. For $m = 2$ and X of small cardinality, μ_R can conveniently be represented as an $n \times n$ matrix where n is the cardinality of X and entry r_{ij} denotes the degree in which R holds between i'th and j'th element of X.

Binary preference relations play a crucial role in the social choice theory. Fuzzy social choice is based on fuzzy binary relations of preference.These can conveniently be interpreted as expressing degrees of preference over pairs of alternatives. Early works elucidating fuzzy preference relations are [4], [5], [21], [23], [17], [6]. We interpret $r_{ij} = 1$ to indicate a definite preference of the ith alternative over the jth one, $r_{ij} = 0$ to indicate a definite preference of the jth alternative over the ith one and $r_{ij} = \frac{1}{2}$ an indifference between the two alternatives.

An $k \times k$ matrix representing a fuzzy preference relation can emerge in many ways. It may be some kind of aggregate of an individual's opinions regarding alternatives if these are evaluated in terms of various criteria of performance. A person looking for a place to live might consider various housing alternatives in terms of price, quality of construction, architectural design, distance from work, etc. Each of these might form the basis of preference relation over the alternatives. By averaging the entries of each preference matrix one might end up with an overall preference matrix over the housing alternatives. Alternatively, the matrix might stand for a fuzzy social preference relation formed by aggregating individual non-fuzzy preferences.

Assuming that the matrix stands for a fuzzy social preference relation we are led to ask how to use it in finding plausible - fuzzy or non-fuzzy - choice sets. Several solution concepts can be suggested:

- the set of α-consensus winners $S_\alpha = \{x_i \mid r_{ij} \geq \alpha, \forall x_j \in X\}$. If the preference relation is reciprocal, i.e. $r_{ij} = 1 - r_{ji}, \forall i \neq j$, and if $\alpha > \frac{1}{2}$, S_α is a singleton.
- the set of minimax consensus winners $S_M = \{x_i \mid r'_i = r'\}$, where $r'_i = min_j\, r_{ij}$ and $r' = max_m\, r'_m$. This set is always nonempty. It is a straightforward generalization of Kramer's minimax set (Ref. 19).
- the set of α-Copeland winners $S_C = \{x_i \mid s^C_i = max_j s^C_j\}$, where $s^C_i =$ $\mid \{x_j \mid r_{ij} \geq \alpha\} \mid$. For each value of $\alpha > \frac{1}{2}$ we get a refinement of the classic Copeland rule.

If the starting point is a set of individual fuzzy preference relations, similar solution concepts can be defined. Thus, for example, the fuzzy α-core $X_\alpha = \{x_j \mid \forall x_i \in X : r_{ij} \leq \alpha$ for at least z individuals$\}$. With $\alpha = \frac{1}{2}$ and z a simple majority, this reduces to the core. Similar extensions can be defined for other standard solution concepts [17]. The literature on fuzzy social choice theory is nowadays vast [11], [3], [9], [10].

Also tournament solutions lend themselves for straightforward extensions. Since tournaments are complete and asymmetric relations, they as such represent a generalization of the usual assumption of choice theory, viz. that the preferences are complete and transitive. A natural way of constructing a tournament is to conduct pairwise comparisons of decision alternatives using the majority rule [18]. Let v_{ij} be the number of individuals preferring x_i to x_j in a pairwise comparison and let v be the total number of individuals. Then a fuzzy $k \times k$ tournament matrix T can be formed by defining the elements of T as follows: $t_{ij} = v_{ij}/v$.

Two important solution concepts in non-fuzzy tournament literature are the uncovered set and the Banks set [2], [14], [16]. An alternative x_i covers another alternative x_j if the former defeats the latter and, moreover, defeats all those alternatives that x_j defeats. A covered alternative will inevitably lose a pairwise majority voting procedure regardless of the order in which the alternatives are brought to pairwise comparisons. Thus, given a profile of individual preferences, an obvious solution concept is the set of uncovered alternatives. However, this set tends to be too large to be useful in choice settings. Hence, various refinements have been suggested. One of them is the Banks set. To define the Banks set one begins with an alternative, say x_1, and finds out whether another alternative exists that defeats it. If there isn't, we are done and conclude that x_1 is the end point of the Banks chain which begins at x_1. If, on the other hand, a x_1-defeating alternative, say x_i, is found, one looks for an alternative that defeats both x_1 and x_i. If no such alternative is found, then x_i is the end point of the Banks chain beginning at x_1. Otherwise one finds out if an alternative defeating all preceding ones - i.e. x_1 and x_i - exists. The search process is continued until one eventually finds no alternative that would defeat all preceding ones in the chain. Inevitably one then reaches an end point of the chain beginning at x_1. Starting from each alternative one necessarily ends up with a chain with an end point. One alternative may, however, give rise to several Banks chains. Now, the Banks set consists of the end points of all Banks chains in the tournament. The main significance of the Banks set is that it coincides with all strategic voting outcomes in binary voting agendas.

Fuzzy analogues of the uncovered set and the Banks set are studied in [18]. In fact, two covering relations, strong and weak, can be defined. The strong covering relation C_s holds between a pair (x_i, x_j) of alternatives if $r_{il} \geq r_{jl}, \forall x_l \in X$ and $r_{ij} > r_{ji}$. The set of strongly uncovered alternatives is thus always a superset of the uncovered set. The weak covering relation C_w , in turn, is defined as follows: $x_i C_w x_j$ if $r_{ij} > r_{ji}$ and $\mid \{x_l \in X \mid r_{il} > r_{jl}\} \mid \;>\; \mid \{x_p \in X \mid r_{jp} > r_{ip}\} \mid, \forall x_l, x_p \in X$.

Obviously, the set of weakly uncovered alternatives is always a subset of the strongly uncovered ones. Moreover, the set of Copeland winners is necessarily within the set of strongly uncovered alternatives which follows from what was said in the preceding paragraph. However, it is not necessary that the Copeland winners are weakly uncovered ones [18].

Introducing outranking information to tournament matrices allows us, thus, to define new solutions to tournaments. These solutions may expand or refine the existing (crisp) ones and, hence, open possibilities for reasonable policy choices in situations where the classic tournament solutions fail, i.e. are either too inclusive or empty. In the same vein as in fuzzy social choice theory one can also take the individual fuzzy preference relations as the point of departure and work out tournament solutions based on them [18].

The main aim of the preceding efforts is to find reasonable choice rules. The main strategy is to introduce more information about individual preferences than is usually the case in the social choice theory.

V. Maximizing Representation

The fuzzy preference apparatus can also be extended to problems of composing multi-member representative bodies (parliaments, committees etc.)[20]. Voter i's preference relation over candidates can be presented as:

$$\begin{array}{cccc} - & r^i_{12} & \dots & r^i_{1k} \\ r^i_{21} & - & \dots & r^i_{2k} \\ \dots & \dots & \dots & \dots \\ r^i_{k1} & r^i_{k2} & \dots & - \end{array}$$

Consider now voter i and a committee c_t consisting of k candidates as required. We are now primarily interested in finding the members of c_t that best represent i. Denote the set of these representatives by $B(i, c_t)$. Several plausible ways of finding the best representatives can be envisioned:

1) $B^i_{\text{sum}}(c_t) = \{j \in c_t | \sum_l r_{jl} \geq \sum_l r_{ql}, \forall q \in c_t\}$,
2) $B^i_{\min}(c_t) = \{j \in c_t | \min_l r_{jl} \geq \min_l r_{ql}, \forall l \in K, \forall q \in c_t\}$,
3) $B^i_h(c_t) = \{j \in c_t | h(j) \geq h(q), \forall q \in c_t\}$ where $h(j) = p\,(\max_l r_{jl}) + (1-p)(\min_l r_{jl})$,
4) $B^i_{\text{cop}}(c_t) = \{j \in c_t | cop(j) \geq cop(q), \forall q \in c_t\}$ where $cop(j) = |\{l \in c_t | r_{jl} > r_{lj}, \forall l \in K\}|$

The first one determines the best representatives on the basis of the sums of the preference degrees obtained by candidates in all pairwise comparisons. This method is very much in the spirit of the Borda count. The second method looks at the minimum preference degree of each candidate when compared with all others and picks the candidate with the largest minimum. It is a variant of the min-max method in social choice theory. The third method is a version of Hurwicz's rule which maximizes the weighted sum of the smallest and largest preference degrees [15]. The fourth method is motivated by Copeland's rule in social choice theory. The Copeland winner is the candidate that defeats more candidates than any other candidate. In the setting of fuzzy preference relation $cop(j)$ is the number of candidates in c_s that are less preferred to j than j is preferred to them. In reciprocal preference matrices, $cop(j)$ is simply the number of entries larger than 0.5 on the j'th row.

Each of these methods singles out the best representatives of every voter in any given committee. Since each of the methods is based on a score, we can define a ranking of candidates in accordance with those scores. From the point of view of representation more important is, however, the ranking over committees ensuing from these methods. The most straightforward way to accomplish this is to define the score of committee c_t as follows:

$$S_t = \sum_{i \in N} \sum_{a \in c_t} \sum_{j \in K} r^i_{aj}.$$

Thus, the score of a committee is the sum of values given by voters to each of its members. The values, in turn, are the sums of preference degrees in all pairwise comparisons. This method is a variation of the Borda count. The most representative committee RC^B would then be:

$$RC^B = \{c_i \in C^k | S_i \geq S_j, \forall c_j \in C^k\}.$$

Although the Chamberlin-Courant approach is very close to the Borda count as well, the above method is not its most plausible fuzzy counterpart [7]. Rather than summing the preference degrees over alternatives and voters, the Chamberlin-Courant approach sums the Borda scores of each voter's representative in any given committee. First we define

$$r^i_j = \sum_{q \in K} r^i_{jq}.$$

Then, for each committee c_t we define:

$$V_{it} = max_{j \in c_t} r^i_j.$$

This can be viewed as the value of the committee c_t to voter i as reflected by the value i assigns to his/her representative in c_t.

Now, the most representative committee in the sense of Chamberlin-Courant is:

$$RC^{CC}_{\text{sum}} = \{c_j \in C^k \mid \sum_i V_{ij} \geq \sum_i V_{iq}, \forall c_q \in C^k, i \in N, j \in K\}.$$

The RC^{CC}_{Sum} committee thus defined is based on the summation of preference degrees in individual preference matrices. In analogous manner one can define the most representative committee in the min-max sense. Let $\underline{r^i_j} = min_{q \in K} r_{jq}$. Now define, for each committee c_t and each voter i:

$$V'_{it} = max_{j \in c_t} \underline{r^i_j}.$$

Then the most representative committee in the min-max sense is:

$$RC'^{CC}_{\text{min}} = \{c_j \in C^k \mid \sum_i V'ij \geq \sum_i V'_{iq}, \forall c_q \in C^k\}.$$

The RC'^{CC}_{min} differs from the previous committee in using the min-max calculus to determine each voter's representative. In a way, RC'^{CC}_{min} mixes two kinds of maximands: the "utilitarian"and "Rawlsian". The former maximizes the average utility, while the latter maximizes the utility of the worst-off individual (Rawls 1971).

A purely Rawlsian committee can also be envisioned. This is obtained as follows:

$$RC^R = \{c_j \in C^k \mid min_i V'_{ij} \geq min_i V'_{iq}, \forall c_q \in C^k\}.$$

In similar vein, one can define Hurwicz and Copeland committees, RC^H and RC^{Co}, respectively. For a fixed value of $p^i \in [0,1]$, let $r_j^{iH} = p^i(\max_q r_{jq}) + (1-p^i)(\min_q r_{jq})$ and $V_{it}^H = max_{j\in c_t} r_j^{iH}$. The set of most representative Hurwicz-type committees is, then:

$$RC^H = \{c_j \in C^k \mid \sum_i V_{ij}^H \geq V_{iq}^H, \forall c_q \in C^k\}.$$

Note that the value p^i is voter specific measure of his/her "optimism", i.e. the weight assigned to $\max_j r_{ij}^i$, i.e. the degree of preference assigned to each candidate in the comparison of its weakest competitor. Intuitively speaking the exclusive emphasis on strongest and weakest pairwise comparisons is somewhat questionable in voting contexts.

To define, the Copeland-type committee, let RC^{Co}, in turn, is based on the voters' value function $r_j^{iCo} =\mid \{q \in K \mid r_{jq} > r_{qj}\} \mid$ and the value function $V_{it}^{iCo} = max_{j\in c_t} r_j^{iCo}$. Now,

$$RC^{Co} = \{c_j \in C^k \mid \sum_i V_{ji}^{iCo} \geq \sum_i V_{qi}^{iCO}, \forall c_q \in C^k\}.$$

Of these four types of committees, the Rawlsian and Copeland types utilize the least amount of the voter preference information. The former looks at the minimal level preference of each candidate when compared with all others. The latter uses only the order information of preference degrees. Of course, if the aim is to economize on information usage, the very idea of resorting to fuzzy preference degrees loses much of its appeal.

VI. CONCLUSION

Individual fuzzy preference relations give rise to a host of choice methods both in settings where single alternative is to be chosen and in contexts where multi-member representative bodies are to be set up. These relations do not, however, make the basic conceptual problems go away. We still need to fix our ideas about what "best" alternatives really mean. Should these be found by looking at alternatives in pairs or should one take a more "holistic" view of the choice situation. The social choice community is today somewhat divided on this issue. For our purposes it is sufficient to note that regardless of which stand on this issue is taken, methods can be devised for solving choice problems – be they single-winner or multiple winner ones. Some of these have been outlined above.

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